

# TENSOR SPACE REPRESENTATIONS OF TEMPERLEY–LIEB ALGEBRA AND GENERALIZED PERMUTATION MATRICES

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## Abstract

Orthogonal projections in  $\mathbb{C}^n \otimes \mathbb{C}^n$  of rank one and rank two that give rise to unitary tensor space representations of the Temperley–Lieb algebra  $TL_N(Q)$  are considered. In the rank one case, a complete classification of solutions is given. In the rank two case, solutions with  $Q$  varying in the ranges  $[2n/3, \infty)$  and  $[n/\sqrt{2}, \infty)$  are constructed for  $n = 3k$  and  $n = 4k$ ,  $k \in \mathbb{N}$ , respectively.

## 1 Introduction

### 1.1 Formulation of the problem and outline of results

Below, we denote by  $M_n$  the ring of  $n \times n$  complex matrices, by  $I_n$  the  $n \times n$  identity matrix, and by  $\otimes$  the Kronecker product.  $\bar{X}$ ,  $X^t$ , and  $X^*$  stand, respectively, for the complex conjugate, the transpose, and the conjugate transpose of  $X \in M_n$ .

In the present article, we will continue the study begun in [2] of a particular class of representations of the Temperley–Lieb algebra  $TL_N(Q)$ . Recall that a unitary representation of  $TL_N(Q)$  on the tensor product space  $(\mathbb{C}^n)^{\otimes N}$  is determined by a matrix  $T \in M_{n^2}$  satisfying the following relations:

$$\begin{aligned} \text{(T1)} \quad & T^* = T, \\ \text{(T2)} \quad & TT = QT, \\ \text{(T3)} \quad & T_{12} T_{23} T_{12} = T_{12}, \\ \text{(T4)} \quad & T_{23} T_{12} T_{23} = T_{23}, \end{aligned}$$

where  $T_{12} \equiv T \otimes I_n$  and  $T_{23} \equiv I_n \otimes T$ . Without a loss of generality, we will always assume that  $Q > 0$ . Apart from  $n$  and  $Q$ , an important parameter of a representation is the rank  $r = \text{rank}(T)$ .

EXAMPLE 1. For  $r = 1$ , the most known solution to (T1)–(T4) is given by

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & \zeta & 0 \\ 0 & \zeta^{-1} & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad q > 0, \quad |\zeta| = 1. \quad (1)$$

The corresponding value of  $Q$  in (T2) is  $Q = q + q^{-1}$ .

The goal of the present article is to consider solutions to (T1)–(T4) in the cases  $r = 1$  and  $r = 2$ . In the latter case, our principal aim is to construct *varying  $Q$  solutions*  $T(q)$ , where  $q$  is a parameter (or a set of parameters) and  $Q = Q(q)$  is a non-constant function of  $q$  (like in Example 1). It should be remarked here that, while rank one solutions to (T1)–(T4) (and their non-Hermitian counterparts) are ubiquitous in the literature, the author is aware of only two examples in the higher rank case — see [1] and [10], where two constructions are given for  $r = n \geq 2$  but in both cases only for  $Q = \sqrt{n}$ .

The paper is organized as follows. In Section 1.2, we reformulate the original problem as a problem of constructing a set of  $r$  matrices  $V_i$  satisfying an orthonormality condition and such that the partitioned matrix  $W_{\mathcal{T}}$  built from them is almost unitary. In Section 2, we give a complete classification of rank one solutions by showing that every suitable matrix  $V \in M_n$  is unitarily congruent to a generalized permutation matrix  $DP_{\sigma}$ , where  $D$  is a non-singular diagonal matrix and  $\sigma$  is an arbitrary involution which has at most one fixed point. In Section 3, we focus on the rank two case, where we have to find a suitable pair  $V_1, V_2 \in M_n$ . In Section 3.1, we establish some properties of  $V_1, V_2$ . In Section 3.2, we construct varying  $Q$  solutions for  $n = 3p$ ,  $p \in \mathbb{N}$  with  $Q \in [2n/3, \infty)$ . In particular, it is shown that every unitary matrix from  $U(2p)$  gives rise to a solution to (T1)–(T4). In Section 3.3 and Section 3.4, we consider the case when a solution is given by (or unitarily congruent to) a pair of generalized permutation matrices, i.e.  $V_1 = D_1 P_{\sigma_1}$ ,  $V_2 = D_2 P_{\sigma_2}$ . In Section 3.3, we establish some necessary conditions for the pair  $\sigma_1, \sigma_2$  and find all suitable pairs for  $n = 4$  and some for  $n > 4$ . All these cases yield solutions with  $Q = n/\sqrt{2}$  and some of them admit varying  $Q$  solutions. In Sections 3.4, we construct varying  $Q$  solutions for  $n = 4l$ ,  $l \in \mathbb{N}$  with  $Q \in [n/\sqrt{2}, \infty)$ . At the end of the section, we briefly discuss the extension of constructed solutions to the non-Hermitian case corresponding to non-unitary representations of  $TL_N(Q)$ . The proofs of all statements are given in the Appendix.

## 1.2 Reformulation of the problem

Let  $\langle, \rangle$  denote the standard inner product on  $\mathbb{C}^n$  and let  $\mathcal{E} = \{e_a\}_{a=1}^n$  be a basis of  $\mathbb{C}^n$  orthonormal w.r.t.  $\langle, \rangle$ . Given a vector  $v \in \mathbb{C}^n \otimes \mathbb{C}^n$ , we will write  $v \sim V \in M_n$  if  $V$  is the matrix of its coefficients, i.e.  $v = \sum_{a,b=1}^n V_{ab} e_a \otimes e_b$ . Similarly, given an  $r$ -dimensional subspace  $\mathcal{T} \subset \mathbb{C}^n \otimes \mathbb{C}^n$ , we will write  $\mathcal{T} \sim \{V_1, \dots, V_r\}$  if the orthonormal set of vectors,  $v_1 \sim V_1, \dots, v_r \sim V_r$ , is a spanning set of  $\mathcal{T}$ . The corresponding orthogonal projection  $P_{\mathcal{T}}$  is represented by the following matrix:

$$P_{\mathcal{T}} = \sum_{s=1}^r \sum_{a,b,c,d=1}^n (V_s)_{ab} (\bar{V}_s)_{cd} E_{ac} \otimes E_{bd}, \quad (2)$$

where  $E_{ab} \in M_n$  is such that  $(E_{ab})_{ij} = \delta_{ai} \delta_{bj}$ .

Every solution to (T1)–(T4) has the form  $T = QP_{\mathcal{T}}$ , where  $P_{\mathcal{T}}$  is given by (2). If  $P_{\mathcal{T}}$  has rank  $r$ , we will say, somewhat abusing the terminology, that the corresponding representation is of rank  $r$ .

EXAMPLE 2. For  $T$  given by (1), we have  $T = (q + q^{-1})P_{\mathcal{T}}$ , where  $\mathcal{T} \sim \{V\}$  and

$$V = \frac{1}{\sqrt{q^2 + 1}} \begin{pmatrix} 0 & \zeta q \\ 1 & 0 \end{pmatrix}, \quad q > 0, \quad |\zeta| = 1. \quad (3)$$

Given a subspace  $\mathcal{T} \sim \{V_1, \dots, V_r\}$  of  $\mathbb{C}^n \otimes \mathbb{C}^n$ , we associate to it the following partitioned matrix  $W_{\mathcal{T}} \in M_{rn}$ :

$$W_{\mathcal{T}} = \sum_{s,m=1}^r E_{sm} \otimes V_m \bar{V}_s. \quad (4)$$

Then we have the following criterion.

**Theorem 1** ([2], Theorem 2).  $T = QP_{\mathcal{T}} \in M_{n^2}$ , where  $Q > 0$  and  $\mathcal{T} \sim \{V_1, \dots, V_r\}$ , is a solution to (T1)–(T4) if and only if  $QW_{\mathcal{T}}$  is a unitary matrix.

Thus, constructing a solution  $T$  to (T1)–(T4) of a rank  $r$  is equivalent to finding  $r$  matrices,  $V_1, \dots, V_r$ , such that the corresponding vectors are orthonormal, i.e.

$$\text{tr}(V_s^* V_m) = \delta_{sm} \quad (5)$$

and the corresponding matrix  $W_{\mathcal{T}}$  is a scalar multiple of a unitary matrix.

It is natural to consider solutions  $T$  and  $T'$  as equivalent if the corresponding sets  $V_1, \dots, V_r$  and  $V'_1, \dots, V'_r$  are related by simultaneous *unitary congruence*:

$$V'_k = g V_k g^t, \quad k = 1, \dots, r, \quad g \in U(n), \quad (6)$$

because, as seen from (2), such  $T$  and  $T'$  are related as follows

$$T' = (g \otimes g) T (g^* \otimes g^*). \quad (7)$$

In this context, it is useful to recall the following criterion of unitary congruence:

**Lemma 1** ([5], Theorem 2.4). *Non-singular matrices  $A, B \in M_n$  are unitarily congruent if and only if there exists a unitary matrix  $g \in U(n)$  such that*

$$A A^* = g B B^* g^*, \quad A \bar{A} = g B \bar{B} g^*. \quad (8)$$

## 2 Representations of rank one

Here we consider solutions to (T1)–(T4) such that  $r \equiv \text{rank}(T) = 1$ .

Let  $V \in M_n$  satisfy the normalization condition

$$\text{tr}(V^* V) = 1. \quad (9)$$

By Theorem 1,  $T = QP_{\mathcal{T}}$ , where  $Q > 0$  and  $\mathcal{T} \sim \{V\}$ , is a solution to (T1)–(T4) if and only if

$$Q V \bar{V} \in U(n), \quad (10)$$

or, equivalently,

$$V \bar{V} V^t V^* = Q^{-2} I_n. \quad (11)$$

REMARK 1. For every  $V$  satisfying (9) and (10), we have  $Q \geq n$  (see Proposition 3 in [2]). The lower bound,  $Q = n$ , is achieved only if  $V$  itself is *almost unitary*, i.e. it is a scalar multiple of a unitary matrix.

REMARK 2. In the rank one case, solving equations (T2)–(T4) without imposing the hermiticity condition (T1) amounts to solving the following counterpart of equation (11):  $VUV^tU^t = Q^{-2}I_n$ , where  $V$  and  $U$  are non-singular matrices such that  $\text{tr}(VU^t) = 1$ . A scheme of construction of suitable pairs  $V, U$  was outlined in [3]. Particular solutions,  $U = Q^{-1}V^{-1}$  and  $U = Q^{-1}(V^t)^{-1}$ , were considered in [9] and [10], respectively. Note that, in the latter case, the only possible value of  $Q$  is  $Q = n$ . This solution is a counterpart of the almost unitary solution to (10) mentioned in Remark 1.

Let us introduce some notations.  $\mathcal{S}_n$  will stand for the symmetric group of degree  $n$ . If we need to write down the explicit form of a permutation  $\sigma \in \mathcal{S}_n$ , we will give its decomposition into cycles. Given an element  $\sigma$  of  $\mathcal{S}_n$ , we will denote by  $P_\sigma \in M_n$  the corresponding permutation matrix, i.e.  $(P_\sigma)_{ij} = \delta_{i, \sigma(j)}$ . Matrix  $P_\sigma^t$  corresponds to  $\sigma^{-1}$ . If  $\sigma$  is an involution, i.e.  $\sigma^{-1} = \sigma$ , then  $P_\sigma$  is a symmetric matrix. Given a diagonal matrix  $D \in M_n$ , we will denote by  $D^\sigma \equiv P_\sigma D P_\sigma^t$  the matrix obtained from  $D$  by the action of the permutation  $\sigma$  on its diagonal entries, i.e. for the matrix entries we have:  $(D^\sigma)_{k,k} = D_{\sigma^{-1}(k), \sigma^{-1}(k)}$ .

The most general form of a solution to equations (9)–(10) is the following.

**Theorem 2.** *Let  $\sigma \in \mathcal{S}_n$  be an involution which has at most one fixed point and  $P_\sigma$  be the corresponding permutation matrix. For every  $V \in M_n$  satisfying (9) and (10), there exists  $g \in U(n)$  such that  $V' = g V g^t$  has the following form:*

$$V' = D P_\sigma, \quad (12)$$

with  $D = \text{diag}(z_1, \dots, z_n)$ , where  $z_k \in \mathbb{C} \setminus \{0\}$  satisfy the following relations:

$$\sum_{k=1}^n |z_k|^2 = 1, \quad z_k z_{\sigma(k)} = Q^{-1} \quad \forall k. \quad (13)$$

In other words, any solution to (9)–(10) is unitarily congruent to a *generalized permutation matrix*  $D P_\sigma$ , where  $P_\sigma$  is an a priori chosen permutation matrix such that

$$P_\sigma^t = P_\sigma, \quad \text{tr } P_\sigma = n \pmod{2}, \quad (14)$$

and the diagonal matrix  $D$  satisfies the following relations:

$$\text{tr } D \bar{D} = 1, \quad D^{-1} = Q D^\sigma. \quad (15)$$

The proof of Theorem 2 is given in the Appendix. Here we remark only that the proof simplifies if the spectrum of  $W_{\mathcal{T}} \equiv V \bar{V}$  is assumed to be simple. In the general case, the proof is based on the results of [4] and [7] on normal forms of *congruence normal* matrices.

Theorem 2 along with equations (6) and (7) allows us to describe all solutions to (T1)–(T4) in the rank one case as follows.

**Corollary 1.** *Let  $\{e_a\}_{a=1}^n$  be the canonical orthonormal basis of  $\mathbb{C}^n$ . For every permutation  $\sigma \in \mathcal{S}_n$  which is an involution and has at most one fixed point and for every  $T \in M_n$  which has rank one and satisfies relations (T1)–(T4) with  $Q > 0$ , there exists a unitary matrix  $g \in U(n)$  such that*

$$T = Q (g \otimes g) (v \otimes v^*) (g^* \otimes g^*), \quad (16)$$

where  $v = \sum_{k=1}^n z_k e_k \otimes e_{\sigma(k)}$  with  $z_k \in \mathbb{C} \setminus \{0\}$  satisfying relations (13).

**REMARK 3.** Consider equations (13) for  $\sigma = (1, n)(2, n-1) \dots$ . Then, given  $z_1, \dots, z_{\lfloor \frac{n}{2} \rfloor}$ , we can obtain the value of  $Q$  and then find the remaining  $z$ 's (up to the sign of  $z_{\frac{n+1}{2}}$  in the case when  $n$  is odd). Thus, a generic solution to (9)–(10) is determined by  $\lfloor \frac{n}{2} \rfloor$  complex parameters. A solution to (9)–(10) which is unitarily congruent to  $V'$  of the form (12), where  $P_\sigma$  does not satisfy one or both conditions (14), will be degenerate, that is, it will have fewer parameters.

**EXAMPLE 3.** For  $n = 2$ , the group  $\mathcal{S}_2$  consists of two elements,  $\sigma = id$  and  $\sigma = (12)$ .  $P_{(12)}$  fulfils conditions (14). So, by Theorem 2, the general solution to (9)–(10) is unitarily congruent to

$$V'(u) = \frac{1}{\sqrt{Q}} \text{diag}(u, u^{-1}) P_{(12)} = \frac{1}{\sqrt{Q}} \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \quad Q = |u|^2 + |u|^{-2}. \quad (17)$$

In accordance with Remark 3, the general solution has one complex parameter,  $u \in \mathbb{C} \setminus \{0\}$ .

Looking for a solution built using  $P_{id}$  instead of  $P_{(12)}$ , we obtain a degenerate solution:

$$V_0 = \frac{1}{\sqrt{Q_0}} \text{diag}(u_0, u_0) P_{id} = \frac{u_0}{\sqrt{Q_0}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_0 = \pm 1, \quad Q_0 = 2. \quad (18)$$

Since Theorem 2 states that every  $n = 2$  solution is unitarily congruent to (17),  $V_0$  must be unitarily congruent to  $V'(u_0) = u_0 P_{(12)} / \sqrt{Q_0}$ . Indeed, Lemma 1 assures that  $P_{(12)}$  and  $P_{id}$  are unitarily congruent. To establish this unitary congruence explicitly, one can verify the following equality:

$$g_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_0^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_0 = \frac{e^{-i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (19)$$

Thus,  $g_0 V'(u_0) g_0^t = V_0$ .

**REMARK 4.**  $V_1, V_2 \in M_n$  that satisfy (9)–(10) for the same value of  $Q$  are not necessary unitarily congruent. Indeed, by Theorem 2, they are unitarily congruent, respectively, to  $V'_1 = D_1 P_\sigma$  and  $V'_2 = D_2 P_\sigma$ , where  $\sigma$  is an involution and  $D_1, D_2$  satisfy (15). Lemma 1 implies that the sets of singular values of unitarily congruent matrices coincide. Thus, a necessary condition for  $V'_1$  and  $V'_2$  (and hence for  $V_1$  and  $V_2$ ) to be unitarily congruent to each other is that  $D_1 \bar{D}_1$  and  $D_2 \bar{D}_2$  coincide up to a permutation of their entries. For  $n \geq 4$ , among solutions to (15) there are pairs  $D_1, D_2$  that do not satisfy this condition.

## 3 Representations of rank two

### 3.1 Preliminary remarks

In the rest of article, we will consider solutions to (T1)–(T4) such that  $r \equiv \text{rank}(T) = 2$ .

Let  $V_1, V_2 \in M_n$  be such that

$$\operatorname{tr}(V_1^* V_1) = \operatorname{tr}(V_2^* V_2) = 1, \quad \operatorname{tr}(V_1^* V_2) = 0. \quad (20)$$

Set

$$W_{\mathcal{T}} \equiv \begin{pmatrix} V_1 \bar{V}_1 & V_2 \bar{V}_1 \\ V_1 \bar{V}_2 & V_2 \bar{V}_2 \end{pmatrix}. \quad (21)$$

By Theorem 1,  $T = QP_{\mathcal{T}}$ ,  $\mathcal{T} \sim \{V_1, V_2\}$ , is a solution to (T1)–(T4) iff  $QW_{\mathcal{T}}$  is a unitary matrix, which is equivalent to the following set of equations:

$$V_1 \bar{V}_1 V_1^t V_1^* + V_2 \bar{V}_1 V_1^t V_2^* = Q^{-2} I_n, \quad (22)$$

$$V_1 \bar{V}_2 V_2^t V_1^* + V_2 \bar{V}_2 V_2^t V_2^* = Q^{-2} I_n, \quad (23)$$

$$V_1 \bar{V}_1 V_2^t V_1^* + V_2 \bar{V}_1 V_2^t V_2^* = 0. \quad (24)$$

It is worth noting that, unlike the rank one case, matrices  $V_1$  and  $V_2$  can be singular.

**Proposition 1.** *If  $V_1, V_2 \in M_n$  satisfy (22)–(24) for some  $Q$ , then  $|\det V_1| = |\det V_2|$ . Furthermore, both  $V_1$  and  $V_2$  are singular if  $n$  is odd.*

REMARK 5. For every pair  $V_1, V_2 \in M_n$  satisfying (20) and (22)–(24), we have

$$Q = \sqrt{2} \text{ if } n = 2, \quad Q \geq \frac{n}{2} \text{ if } n \geq 3, \quad (25)$$

see Theorem 3 and Corollary 3 in [2]. Furthermore,  $Q = n/\sqrt{2}$  if either  $V_1$  or  $V_2$  is a scalar multiple of a unitary matrix, cf. Proposition 6 in [2].

The condition that  $QW_{\mathcal{T}}$  be unitary implies that each block  $QV_i \bar{V}_j$  is a contraction. If at least one of the blocks is itself a scalar multiple of a unitary matrix, then the estimate (25) sharpens as follows.

**Proposition 2.** *Let  $V_1, V_2 \in M_n$  satisfy (22)–(24) for some  $Q > 0$ . Suppose, in addition, that  $\alpha V_1 \bar{V}_1$  is unitary for some  $\alpha > 0$  and  $\operatorname{tr}(V_1^* V_1) = 1$ . Then*

i)  $\alpha V_2 \bar{V}_1$ ,  $\alpha V_1 \bar{V}_2$ , and  $\alpha V_2 \bar{V}_2$  are unitary.

ii) There exist  $g, g' \in U(n)$  such that  $V_2 = V_1 g$  and  $V_2 = g' V_1$ .

iii)  $V_1, V_2$  satisfy (20).

iv)  $\alpha = \sqrt{2}Q$  and

$$Q \geq \frac{n}{\sqrt{2}}. \quad (26)$$

Examples of rank two solutions, where all the blocks of  $QW_{\mathcal{T}}$  are almost unitary, will be given in Sections 3.3 and 3.4

### 3.2 Solutions for $n = 3p$

Here we will construct rank two solutions in the case when  $n$  is a multiple of 3. Consider the following ansatz:

$$V_1 = \begin{pmatrix} 0 & F_{11} & 0 \\ \bar{G}_{11} & 0 & \bar{G}_{12} \\ 0 & F_{21} & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & F_{12} & 0 \\ \bar{G}_{21} & 0 & \bar{G}_{22} \\ 0 & F_{22} & 0 \end{pmatrix}, \quad (27)$$

where  $F_{ij}, G_{ij} \in M_p$ ,  $p \geq 1$ . Note that  $\det V_1 = \det V_2 = 0$ , so that the ansatz is consistent with Proposition 1 for all  $p$ .

**Theorem 3.** *For  $p \in \mathbb{N}$ , let  $\alpha_1, \alpha_2$  be some positive numbers such that*

$$\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} = \frac{1}{p}. \quad (28)$$

*Suppose that  $F_{ij}, G_{ij} \in M_p$  are such that the following partitioned matrices*

$$H_1 = \alpha_1 \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad H_2 = \alpha_2 \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \quad (29)$$

*are unitary. If  $p > 1$ , suppose, in addition, that the equality*

$$\zeta (G_{11} F_{12} + G_{12} F_{22}) = G_{21} F_{11} + G_{22} F_{21} \quad (30)$$

*holds for some  $\zeta \in \mathbb{C}$  such that  $|\zeta| = 1$ .*

*Then  $V_1, V_2 \in M_{3p}$  given by (27) satisfy relations (20) and (22)–(24) with*

$$Q = \alpha_1 \alpha_2, \quad (31)$$

*and, therefore,  $T = QP_T$ ,  $\mathcal{T} \sim \{V_1, V_2\}$  is a solution to (T1)–(T4).*

REMARK 6. Condition (28) implies the following inequality for  $Q$  given by (31):

$$Q \geq 2p = \frac{2}{3}n. \quad (32)$$

**Proposition 3.** *Let  $p \in \mathbb{N}$  and let positive  $\alpha_1, \alpha_2$  satisfy (28).*

*i) The hypotheses of Theorem 3 are fulfilled if*

$$\alpha_1 F_{ij} = U_{ij}, \quad \alpha_2 G_{ij} = U_{ji}^*, \quad (33)$$

*providing that the partitioned matrix  $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$  is unitary.*

*ii) The hypotheses of Theorem 3 are fulfilled if*

$$F_{12} = -w F_{22}, \quad F_{21} = \bar{w} F_{11}, \quad G_{12} = w G_{11}, \quad G_{21} = -\bar{w} G_{22}, \quad w \in \mathbb{C}, \quad (34)$$

*providing that  $\beta_1 F_{11}, \beta_1 F_{22}, \beta_2 G_{11}, \beta_2 G_{22}$  are unitary for  $\beta_i = \alpha_i \sqrt{1 + |w|^2}$ .*

EXAMPLE 4. The pair

$$V_1 = \begin{pmatrix} 0 & z_1 & 0 \\ z_2 & 0 & \bar{w}z_2 \\ 0 & \bar{w}z_1 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & -\zeta_1 w z_1 & 0 \\ -\zeta_2 w z_2 & 0 & \zeta_2 z_2 \\ 0 & \zeta_1 z_1 & 0 \end{pmatrix}, \quad (35)$$

where  $z_1, z_2, w, \zeta_1, \zeta_2 \in \mathbb{C}$  and

$$|\zeta_1| = |\zeta_2| = 1, \quad |z_1||z_2| \neq 0, \quad (|z_1|^2 + |z_2|^2)(1 + |w|^2) = 1, \quad (36)$$

satisfies (20) and (22)–(24) with

$$Q^{-1} = |z_1||z_2|(1 + |w|^2). \quad (37)$$

In particular, setting  $w = 0$ ,  $\zeta_2 = -\zeta_1 = 1$ , and  $z_1 = (q^4 + 1)^{-\frac{1}{2}}$ ,  $z_2 = q^2 z_1$ ,  $q > 0$ , we recover Example 13 from [2] constructed as a TL pair for the quantum algebra  $U_q(su_2)$ .

REMARK 7. For  $p = 1$ , condition (30) follows from the hypothesis that  $H_1, H_2$  are unitary (indeed, if  $A = H_2 H_1 \in U(2)$ , then  $|A_{12}| = |A_{21}|$  which is equivalent to (30)). Therefore, taking two generic elements from  $U(2)$  as  $H_1, H_2$ , we obtain the following solution.

EXAMPLE 5. The pair

$$V_1 = \begin{pmatrix} 0 & z_1 & 0 \\ z_2 & 0 & z_3 \\ 0 & z_4 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & -\zeta_1 \bar{z}_4 & 0 \\ -\zeta_2 \bar{z}_3 & 0 & \zeta_2 \bar{z}_2 \\ 0 & \zeta_1 \bar{z}_1 & 0 \end{pmatrix}, \quad (38)$$

where  $z_1, z_2, z_3, z_4, \zeta_1, \zeta_2 \in \mathbb{C}$  and

$$|\zeta_1| = |\zeta_2| = 1, \quad |z_1| + |z_4| \neq 0, \quad |z_2| + |z_3| \neq 0, \quad |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1, \quad (39)$$

satisfies (20) and (22)–(24) with

$$Q^{-2} = (|z_1|^2 + |z_4|^2)(|z_2|^2 + |z_3|^2). \quad (40)$$

Employing generalized permutation matrices, we will construct a solution generalizing Example 5 for which (30) holds non-trivially (that is, unlike for the cases given in Proposition 3, the l.h.s. and the r.h.s. of (30) do not vanish identically).

**Proposition 4.** *Given  $p \in \mathbb{N}$  and  $\sigma_1, \sigma_2 \in \mathcal{S}_p$ , let  $P_{\sigma_1}, P_{\sigma_2}$  be the corresponding permutation matrices and let  $F_{ij}, G_{ij} \in M_p$  be given by*

$$F_{11} = P_{\sigma_1} D_1, \quad F_{12} = -P_{\sigma_1} \bar{D}_4 Z_1, \quad F_{21} = P_{\sigma_1} D_4, \quad F_{22} = P_{\sigma_1} \bar{D}_1 Z_1, \quad (41)$$

$$G_{11} = D_2 P_{\sigma_2}, \quad G_{12} = D_3 P_{\sigma_2}, \quad G_{21} = -Z_2 \bar{D}_3 P_{\sigma_2}, \quad G_{22} = Z_2 \bar{D}_2 P_{\sigma_2}, \quad (42)$$

where  $D_i \in M_p$  are diagonal matrices and  $Z_i \in U(p)$  are diagonal unitary matrices. Then the hypotheses of Theorem 3 are satisfied providing that

$$\alpha_1^2 (D_1 \bar{D}_1 + D_4 \bar{D}_4) = I_p = \alpha_2^2 (D_2 \bar{D}_2 + D_3 \bar{D}_3) \quad (43)$$

for some positive  $\alpha_1, \alpha_2$  satisfying (28) and

$$\zeta M Z_1^{\sigma_2 \circ \sigma_1} = Z_2 \bar{M}, \quad (44)$$

where  $M \equiv (D_3 \bar{D}_1^{\sigma_2 \circ \sigma_1} - D_2 \bar{D}_4^{\sigma_2 \circ \sigma_1})$  and  $|\zeta| = 1$ .



### 3.3 Generalized permutations matrices, solutions for $Q = n/\sqrt{2}$

For  $n$  even, we will look for solutions to (22)–(24) that can be brought by a simultaneous unitary congruence (6) to a pair of generalized permutation matrices,

$$V_1 = D_1 P_{\sigma_1}, \quad V_2 = D_2 P_{\sigma_2}, \quad (45)$$

where  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  and  $D_1, D_2 \in M_n$  are non-singular diagonal matrices. Since pairs related by (6) are regarded as equivalent and all permutation matrices are unitary, we will search for pairs of the form (45) up to the transformations

$$\sigma_1 \rightarrow \tau \circ \sigma_1 \circ \tau^{-1}, \quad \sigma_2 \rightarrow \tau \circ \sigma_2 \circ \tau^{-1}, \quad \tau \in \mathcal{S}_n. \quad (46)$$

For  $V_1, V_2$  of the form (45), conditions (20) turn into

$$\text{tr } D_1 \bar{D}_1 = \text{tr } D_2 \bar{D}_2 = 1, \quad (47)$$

$$\text{tr}(D_1 \bar{D}_2 P_{\sigma_1} P_{\sigma_2}^t) = 0, \quad (48)$$

and equations (22)–(24) are equivalent to the following system:

$$D_1 \bar{D}_1 D_1^{\sigma_1} \bar{D}_1^{\sigma_1} + D_2 \bar{D}_2 D_1^{\sigma_2} \bar{D}_1^{\sigma_2} = Q^{-2} I_n, \quad (49)$$

$$D_1 \bar{D}_1 D_2^{\sigma_1} \bar{D}_2^{\sigma_1} + D_2 \bar{D}_2 D_2^{\sigma_2} \bar{D}_2^{\sigma_2} = Q^{-2} I_n, \quad (50)$$

$$D_1 \bar{D}_1^{\sigma_1} P_{\sigma'} D_2^{\sigma_1} \bar{D}_1 + D_2 \bar{D}_1^{\sigma_2} P_{\sigma''} D_2^{\sigma_2} \bar{D}_2 = 0, \quad (51)$$

where  $\sigma' = \sigma_1 \circ \sigma_1 \circ \sigma_2^{-1} \circ \sigma_1^{-1}$  and  $\sigma'' = \sigma_2 \circ \sigma_1 \circ \sigma_2^{-1} \circ \sigma_2^{-1}$ . Since  $D_1, D_2$  are non-singular, equation (51) requires that  $\sigma' = \sigma''$  that is

$$\sigma_2^{-1} \circ \sigma_1 \circ \sigma_1 \circ \sigma_2^{-1} = \sigma_1 \circ \sigma_2^{-1} \circ \sigma_2^{-1} \circ \sigma_1. \quad (52)$$

Below, we will write  $\sigma_1 \asymp \sigma_2$  if  $\sigma_1$  and  $\sigma_2$  are commuting permutations. For instance, (52) is equivalent to the condition  $\sigma_1 \circ \sigma_2^{-1} \asymp \sigma_2^{-1} \circ \sigma_1$ .

For  $n$  even, let us call  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  an *admissible* pair of permutations if a) they satisfy relation (52); b) they have no common fixed points; c)  $\sigma_2^{-1} \circ \sigma_1$  has no fixed points if  $\sigma_1$  and  $\sigma_2$  are involutions or if  $\sigma_1 \asymp \sigma_2$ . Two admissible pairs are regarded as equivalent if they are related by the transformation (46) combined, if necessary, with the exchange  $\sigma_1 \leftrightarrow \sigma_2$ .

**Lemma 2.** *If a pair  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  is not admissible, then equation (51) has no solution for non-singular diagonal matrices  $D_1, D_2 \in M_n$ .*

Note that Lemma 2 excludes, in particular, the case  $\sigma_1 = \sigma_2$ .

**REMARK 8.** If  $\sigma_2^{-1} \circ \sigma_1$  has no fixed points, then condition (48) is satisfied for any  $D_1, D_2$ .

**Proposition 5.** *i) Every admissible pair  $\sigma_1, \sigma_2 \in \mathcal{S}_4$  is equivalent to one of the pairs in the following list:*

- |  |  |
|--|--|
| a) $\sigma_1 = id, \sigma_2 = (12)(34);$           | b) $\sigma_1 = id, \sigma_2 = (1234);$         |
| c) $\sigma_1 = (1)(23)(4), \sigma_2 = (14)(2)(3);$ | d) $\sigma_1 = (1)(23)(4), \sigma_2 = (1342);$ |
| e) $\sigma_1 = (1)(23)(4), \sigma_2 = (12)(34);$   | f) $\sigma_1 = (1)(234), \sigma_2 = (321)(4);$ |
| g) $\sigma_1 = (1234), \sigma_2 = (13)(24);$       | h) $\sigma_1 = (1234), \sigma_2 = (12)(34);$   |
| i) $\sigma_1 = (1234), \sigma_2 = (4321);$         | j) $\sigma_1 = (12)(34), \sigma_2 = (14)(23);$ |

ii) For every admissible pair  $\sigma_1, \sigma_2$  in this list, there exist vectors  $\vec{u}, \vec{v} \in \mathbb{R}^4$  such that matrices  $V_1 = \frac{1}{2} \text{diag}(e^{i\pi u_1}, \dots, e^{i\pi u_4}) P_{\sigma_1}$  and  $V_2 = \frac{1}{2} \text{diag}(e^{i\pi v_1}, \dots, e^{i\pi v_4}) P_{\sigma_2}$  satisfy (20) and (22)–(24) with  $Q = 2\sqrt{2}$ .

REMARK 9.  $\sigma_2^{-1} \circ \sigma_1$  has no fixed points for all the pairs in the list except the case h).

Let us say that  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  have complementary sets of fixed points if for every  $k = 1, \dots, n$ , we have either  $\sigma_1(k) = k$  and  $\sigma_2(k) \neq k$  or  $\sigma_2(k) = k$  and  $\sigma_1(k) \neq k$ . The cases a), b), and c) are of this type and they admit the following generalization.

**Proposition 6.** For  $n$  even, let  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  be composed only of 1-cycles (corresponding to fixed points) and cycles of even length. If such  $\sigma_1, \sigma_2$  have complementary sets of fixed points, then  $\sigma_1, \sigma_2$  is an admissible pair and there exist diagonal matrices  $D_1, D_2 \in M_n$  such that  $V_1 = D_1 P_{\sigma_1}$  and  $V_2 = D_2 P_{\sigma_2}$  satisfy (20) and (22)–(24) with  $Q = n/\sqrt{2}$ .

We will see below that the cases h), i), and j) and their generalizations to greater  $n$  divisible by four allow us to construct representations of rank two not only for  $Q = n/\sqrt{2}$  but for  $Q$  varying in the range  $[n/\sqrt{2}, \infty)$ .

### 3.4 Generalized permutations matrices, varying $Q$ solutions for $n = 4l$

Observe that relation (52) holds if  $\sigma_1, \sigma_2$  satisfy the following conditions:

$$\sigma_1 \asymp \sigma_2 \circ \sigma_2 \quad \text{and} \quad \sigma_2 \asymp \sigma_1 \circ \sigma_1. \quad (53)$$

For such  $\sigma_1, \sigma_2$ , we have

$$\sigma' = \sigma'' = \sigma_2^{-1} \circ \sigma_1, \quad (54)$$

and equation (51) acquires the following form:

$$D_1^{\sigma_2} \bar{D}_1^{\sigma_2 \circ \sigma_1} D_2^{\sigma_1 \circ \sigma_1} \bar{D}_1^{\sigma_1} = -D_2^{\sigma_2} \bar{D}_1^{\sigma_2 \circ \sigma_2} D_2^{\sigma_1 \circ \sigma_2} \bar{D}_2^{\sigma_1}. \quad (55)$$

Note that (53) holds, in particular, if  $\sigma_1$  and  $\sigma_2$  commute or if they both are involutions.

EXAMPLE 6. For  $n$  even, the following pairs  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  satisfy (53):

$$\sigma_1 = (1, \dots, n), \quad \sigma_2 = (n, \dots, 1), \quad (56)$$

$$\sigma_1 = (1n)(2, n-1) \dots \left(\frac{n}{2}, \frac{n}{2} + 1\right), \quad \sigma_2 = (12)(34) \dots (n-1, n). \quad (57)$$

They are admissible, respectively, for  $n = 2l + 2$ ,  $l \in \mathbb{N}$  and  $n = 4l$ ,  $l \in \mathbb{N}$ . For  $n = 4$ , (56) and (57) recover, respectively, the cases i) and j) in Proposition 5.

**Theorem 4.** For  $n$  even, let  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  satisfy (53) and let  $\sigma_2^{-1} \circ \sigma_1$  have no fixed points. Let  $P_{\sigma_1}, P_{\sigma_2}$  be the corresponding permutation matrices and let  $A, B \in M_n$  be given by

$$A = (I_n + P_{\sigma_2})(P_{\sigma_2} - P_{\sigma_1}), \quad B = (I_n + P_{\sigma_1})(P_{\sigma_1} - P_{\sigma_2}). \quad (58)$$

Let  $\vec{x} \in \mathbb{R}^n$  be a vector such that

$$P_{\sigma_1} \vec{x} = P_{\sigma_2} \vec{x} = -\vec{x}. \quad (59)$$

Suppose that there exist vectors  $\vec{u}, \vec{v} \in \mathbb{R}^n$  such that all the components of the vector

$$\vec{w} = A\vec{u} + B\vec{v} \quad (60)$$

are odd integers.

Let  $D_1, D_2 \in M_n$  be diagonal matrices such that

$$(D_1)_{kk} = \mu^{-1} e^{x_k + i\pi u_k}, \quad (D_2)_{kk} = \mu^{-1} e^{x_k + i\pi v_k}, \quad (61)$$

where  $\mu^2 = \sum_{k=1}^n e^{2x_k}$ .

Then  $V_1 = D_1 P_{\sigma_1}, V_2 = D_2 P_{\sigma_2}$  satisfy relations (20) and (22)–(24) with  $Q$  given by

$$Q = \frac{1}{\sqrt{2}} \sum_{k=1}^n e^{2x_k}, \quad (62)$$

and, therefore,  $T = QP_{\mathcal{T}}, \mathcal{T} \sim \{V_1, V_2\}$  is a solution to (T1)–(T4).

REMARK 10. Condition (59) implies that  $D_i \bar{D}_i^{\sigma_i}$  is a multiple of a unitary matrix and hence so is  $V_i \bar{V}_i$ . Therefore, by Proposition 2, we have  $Q \geq n/\sqrt{2}$ . The value  $Q = n/\sqrt{2}$  is achieved only if  $\vec{x} = \vec{0}$ , in which case matrices  $V_1, V_2$  are themselves almost unitary.

For the admissible pairs given in Example 6, vector  $\vec{x} = (x, -x, x, -x, \dots)$  satisfies (59) and, moreover, condition (60) turns out to be resolvable if  $n$  is divisible by four. Let us write  $D = \text{diag}_m(d_1, \dots, d_m)$  if  $D$  is a diagonal matrix such that  $(D)_{k+m, k+m} = (D)_{k, k}$ .

**Proposition 7.** For  $n = 4l, l \in \mathbb{N}$ , let  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  be given by either (56) or (57) and let  $D_1, D_2 \in M_n$  be given by

$$D_1 = \text{diag}_4(z_1, z_2, z_1, z_2), \quad D_2 = \text{diag}_4(z_1, -\zeta z_2, z_1, \zeta z_2), \quad (63)$$

where  $z_1, z_2, \zeta \in \mathbb{C}$  are such that

$$|\zeta| = 1, \quad |z_1||z_2| \neq 0, \quad |z_1|^2 + |z_2|^2 = \frac{2}{n}. \quad (64)$$

Then the pair  $V_1 = D_1 P_{\sigma_1}, V_2 = D_2 P_{\sigma_2}$  satisfies (20) and (22)–(24) with

$$Q = \frac{1}{\sqrt{2}|z_1||z_2|}. \quad (65)$$

EXAMPLE 7. For  $n = 4$ ,  $V_1, V_2$  corresponding to  $\sigma_1, \sigma_2$  given by (56) look as follows:

$$V_1 = \begin{pmatrix} 0 & 0 & 0 & z_1 \\ z_2 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & z_1 & 0 & 0 \\ 0 & 0 & -\zeta z_2 & 0 \\ 0 & 0 & 0 & z_1 \\ \zeta z_2 & 0 & 0 & 0 \end{pmatrix} \quad (66)$$

and  $V_1, V_2$  corresponding to  $\sigma_1, \sigma_2$  given by (57) are

$$V_1 = \begin{pmatrix} 0 & 0 & 0 & z_1 \\ 0 & 0 & z_2 & 0 \\ 0 & z_1 & 0 & 0 \\ z_2 & 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & z_1 & 0 & 0 \\ -\zeta z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1 \\ 0 & 0 & \zeta z_2 & 0 \end{pmatrix} \quad (67)$$

In both cases,  $|z_1|^2 + |z_2|^2 = 1/2$  and  $|z_1||z_2| \neq 0$ .

A pair  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  can be admissible despite that  $\sigma_2^{-1} \circ \sigma_1$  has fixed points. The case h) in Proposition 5 is an example of such a pair. It can be generalized as follows.

EXAMPLE 8. For  $n = 2l + 2$ ,  $l \in \mathbb{N}$ , the following pair  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  is admissible and satisfies (53):

$$\sigma_1 = (1, \dots, n), \quad \sigma_2 = (12)(34) \dots (n-1, n). \quad (68)$$

**Proposition 8.** For  $n = 4l$ ,  $l \in \mathbb{N}$ , let  $\sigma_1, \sigma_2 \in \mathcal{S}_n$  be given by (68) and let  $D_1, D_2 \in M_n$  be given by

$$D_1 = \text{diag}_4(z_1, z_2, z_1, z_3), \quad D_2 = \text{diag}_4(z_1, -\zeta \bar{z}_3, z_1, \zeta \bar{z}_2), \quad (69)$$

where  $z_1, z_2, z_3, \zeta \in \mathbb{C}$  are such that

$$|\zeta| = 1, \quad |z_1| \neq 0, \quad |z_2| + |z_3| \neq 0, \quad 2|z_1|^2 + |z_2|^2 + |z_3|^2 = \frac{4}{n}. \quad (70)$$

Then the pair  $V_1 = D_1 P_{\sigma_1}$ ,  $V_2 = D_2 P_{\sigma_2}$  satisfies (20) and (22)–(24) with

$$Q = \frac{1}{|z_1| \sqrt{|z_2|^2 + |z_3|^2}}. \quad (71)$$

EXAMPLE 9. For  $n = 4$ ,  $V_1, V_2$  look as follows:

$$V_1 = \begin{pmatrix} 0 & 0 & 0 & z_1 \\ z_2 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_3 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & z_1 & 0 & 0 \\ -\zeta \bar{z}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1 \\ 0 & 0 & \zeta \bar{z}_2 & 0 \end{pmatrix}, \quad (72)$$

where  $2|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$  and  $|z_1|(|z_2| + |z_3|) \neq 0$ .

REMARK 11. In Proposition 8,  $V_1 \bar{V}_1$  and  $V_2 \bar{V}_2$  are not multiples of unitary matrices unless  $|z_2| = |z_3|$ . Furthermore, unlike the case of Proposition 7, we can set either  $z_2 = 0$  or  $z_3 = 0$ . In which case, both  $V_1$  and  $V_2$  become degenerate in accordance with Proposition 1.

REMARK 12. Despite that  $V_i \bar{V}_i$  in Proposition 8 are not in general almost unitary,  $Q$  given by (71) satisfies the same inequality,  $Q \geq n/\sqrt{2}$ , as in the case of Proposition 7.

REMARK 13. The three pairs of matrices  $V_1, V_2 \in M_{4l}$  constructed in Proposition 7 and Proposition 8 are not unitarily congruent to each other for generic values of  $z_1, z_2, z_3$ . Indeed,  $\chi(V) \equiv \text{tr}(V \bar{V})$  is invariant under unitary congruence. But for generic  $z_1, z_2, z_3$ , we have  $\chi(V_1) = \chi(V_2) = 0$  if  $\sigma_1, \sigma_2$  are given by (56),  $\chi(V_1) \neq 0$ ,  $\chi(V_2) \neq 0$  if  $\sigma_1, \sigma_2$  are given by (57), and  $\chi(V_1) = 0$ ,  $\chi(V_2) \neq 0$  if  $\sigma_1, \sigma_2$  are given by (68).

REMARK 14. It was pointed out in Remark 1 of [2] that certain varying  $Q$  solutions to (T1)–(T4) extend to non-Hermitian solutions to (T2)–(T4) thus extending corresponding unitary tensor space representations of  $TL_N(Q)$  to non-unitary ones. For instance,  $T$  given in Example 1 remains a solution to (T2)–(T4) for  $q, \zeta \in \mathbb{C} \setminus \{0\}$ . Below, we give an example for the rank two case.

EXAMPLE 10. For  $n = 4l$ ,  $l \in \mathbb{N}$  and the representations constructed in Proposition 7, parametrize  $z_1$  and  $z_2$  as follows (cf. equation (3)):

$$z_1 = \sqrt{\frac{2}{n}} \frac{q \xi_1}{\sqrt{q^2 + 1}}, \quad z_2 = \sqrt{\frac{2}{n}} \frac{\xi_2}{\sqrt{q^2 + 1}}, \quad q > 0, \quad |\xi_1| = |\xi_2| = 1. \quad (73)$$

By (65), we have  $Q = n\sqrt{2}(q+q^{-1})/4$  and it is evident from (2) that entries of  $T(q, \xi_1, \xi_1, \zeta) = QP_{\mathcal{T}}$  are rational functions in  $q, \xi_1, \xi_2, \zeta$  with a pole at the origin of the complex plain. Therefore, equalities (T2)–(T4) imply that certain rational functions in these variables vanish identically and hence these equalities remain valid for  $q, \xi_1, \xi_2, \zeta \in \mathbb{C} \setminus \{0\}$ .

The representations constructed in Proposition 8 extend to non-unitary ones in the same vein.

## Appendix

The proof of Theorem 2 will be preceded by the following Lemma.

**Lemma 3.** *If  $V \in M_n$  satisfies (10) with  $Q > 0$  and  $W_{\mathcal{T}} \equiv V\bar{V}$ , then the following holds:*

- i)  $V$  is non-singular.  $\det(QW_{\mathcal{T}}) = 1$ .*
- ii) The set of singular values of  $V$  comprises  $\lfloor \frac{n}{2} \rfloor$  pairs of the form  $(\lambda_k, \lambda'_k)$ , where  $\lambda_k \lambda'_k = Q^{-1}$ , and if  $n$  is odd, one unpaired singular value equal to  $Q^{-\frac{1}{2}}$ .*
- iii) The set of eigenvalues of  $QW_{\mathcal{T}}$  comprises  $\lfloor \frac{n}{2} \rfloor$  pairs of the form  $(\zeta_k, \bar{\zeta}_k)$ , where  $|\zeta_k| = 1$ , and if  $n$  is odd, one unpaired eigenvalue equal to unity.*

**Proof of Lemma 3.** *i)* We have  $\det(QW_{\mathcal{T}}) = Q^n |\det V|^2$  and also  $|\det(QW_{\mathcal{T}})| = 1$  since  $QW_{\mathcal{T}}$  unitary. Hence  $\det V \neq 0$  and  $\det(QW_{\mathcal{T}}) = 1$  because  $Q > 0$ .

*ii)* Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the set of singular values of  $V$ . Then the set of eigenvalues of  $V^*V$  and  $\overline{VV^*}$  is  $\{\lambda_1^2, \dots, \lambda_n^2\}$ . Equation (11) can be rewritten in the form  $Q^2 \overline{VV^*} = (V^*V)^{-1}$ . Which implies that  $Q^2 \{\lambda_1^2, \dots, \lambda_n^2\}$  coincides with  $\{\lambda_n^{-2}, \dots, \lambda_1^{-2}\}$ , i.e.  $\lambda_k \lambda_{n+1-k} = Q^{-1}$ . If  $n$  is odd, we have  $\lambda_{(n+1)/2}^2 = Q^{-1}$ .

*iii)* Note that  $\bar{W}_{\mathcal{T}} = \bar{V}V = V^{-1}W_{\mathcal{T}}V$ . Hence, if  $\zeta \neq \pm 1$  is an eigenvalue of  $QW_{\mathcal{T}}$  and the corresponding eigenspace is spanned by vectors  $x_1, \dots, x_m$ , then the eigenspace corresponding to  $\bar{\zeta}$  is spanned by  $V\bar{x}_1, \dots, V\bar{x}_m$ . It remains to note that the eigenspace corresponding to  $\zeta = -1$  is even-dimensional because, by *i)*,  $\det(QW_{\mathcal{T}}) = 1$ .  $\square$

**Proof of Theorem 2.** Let  $V \in M_n$  satisfy (9) and (10) and let  $W_{\mathcal{T}} \equiv V\bar{V}$ .

Let us first prove Theorem 2 assuming that the spectrum of  $W_{\mathcal{T}}$  is simple. In this case, taking Lemma 3 into account, it follows that there exists  $g \in U(n)$  such that  $\tilde{W} = QgW_{\mathcal{T}}g^*$  is a diagonal unitary matrix such that

$$\tilde{W}_{aa} \tilde{W}_{bb} = 1 \quad \text{if and only if} \quad b = \sigma(a), \quad (74)$$

where  $\sigma \in \mathcal{S}_n$  is an involution that has no fixed points if  $n$  is even and one fixed point if  $n$  is odd. Note that  $\tilde{W}^\sigma = \tilde{W}^{-1}$ . For  $V_0 = gVg^t$ , we have

$$QV_0\bar{V}_0 = \tilde{W}, \quad Q\bar{V}_0V_0 = \tilde{W}^{-1}. \quad (75)$$

Whence we conclude that  $V_0 = \tilde{W} V_0 \tilde{W}$  and thus  $(V_0)_{ab} = \tilde{W}_{aa} \tilde{W}_{bb} (V_0)_{ab}$ . Taking (74) into account, we infer that  $(V_0)_{ab} = 0$  unless  $b = \sigma(a)$ . That is, we have established that  $V$  is unitarily congruent to  $V_0 = D_0 P_\sigma$ , where  $D_0$  is a diagonal matrix which, by (75), satisfies the equation  $Q D_0 \tilde{D}_0^\sigma = \tilde{W}$ . The general solution to this equation is  $D_0 = T w H$ , where  $T = \text{diag}(t_1, t_2, \dots)$ ,  $t_k > 0$  and  $H = \text{diag}(\xi_1, \xi_2, \dots)$ ,  $|\xi_k| = 1$  are such that  $Q T^\sigma = T^{-1}$  and  $H^\sigma = H$ , whereas  $w$  is a diagonal unitary matrix such that  $w^2 = \tilde{W}$  and  $w^\sigma = w^{-1}$ . Let  $h$  be a unitary diagonal matrix such that  $h^2 = H$  and  $h^\sigma = h$ . Then  $V'_0 = h^{-1} V_0 h^{-1}$  is unitarily congruent to  $V_0$  (and hence to  $V$ ) and we have  $V'_0 = D P_\sigma$ , where  $D = T w$ . Observe that  $Q D D^\sigma = Q T w T^\sigma w^\sigma = I_n$ . Thus,  $D$  satisfies the second equation in (15). The first equation in (15) for  $D$  follows from the condition (9).

In order to prove Theorem 2 in the general case, i.e. without assuming anything about the spectrum of  $W_{\mathcal{T}}$ , we will invoke some results about *congruence normal* matrices obtained first in [4] and developed further in [7] (see also [8] and Problem 4.4.P41 in [6]).

Recall that  $A \in M_n$  is called congruence normal if  $A\bar{A}$  is normal.

**Lemma 4** ([7], Theorem 5.3). *If  $A$  is a non-singular congruence normal matrix, then  $A$  is unitarily congruent to a block-diagonal matrix, where each block is of the form*

$$(s) \quad \text{or} \quad \begin{pmatrix} 0 & \mu t \\ t & 0 \end{pmatrix}, \quad s, t > 0, \mu \in \mathbb{C} \setminus \{0, 1\}. \quad (76)$$

REMARK 15. For  $\mu = 1$ , the  $2 \times 2$  block in (76) is unitarily congruent to  $tI_2$ , cf. eq. (19).

Now, let  $V \in M_n$  satisfy (10) and  $A \equiv Q^{\frac{1}{2}} V$ . Since the r.h.s. of (10) is a unitary matrix,  $A$  is non-singular and congruence normal. Therefore, by Lemma 4,  $A$  is unitarily congruent to a block-diagonal matrix with blocks as in (76). Hence  $A\bar{A}$  is unitarily similar to a block-diagonal matrix with blocks  $(s_k^2)$  and  $t_k^2 \text{diag}(\mu_k, \bar{\mu}_k)$ . Note that  $s_k = |\mu_k| t_k^2 = 1$  because  $A\bar{A}$  is unitary. So,  $A$  is unitarily congruent to  $A' = \text{diag}(1, \dots, 1, B_1, B_2, \dots)$ , where  $B_k = \begin{pmatrix} 0 & \zeta_k/t_k \\ t_k & 0 \end{pmatrix}$ ,  $t_k > 0$ ,  $|\zeta_k| = 1$ . By Lemma 1,  $\text{diag}(1, 1)$  and  $B_k$  are unitarily congruent, respectively, to  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $\tilde{B}(z_k) = \begin{pmatrix} 0 & z_k \\ z_k^{-1} & 0 \end{pmatrix}$ , where  $z_k = \zeta_k^{1/2}/t_k \in \mathbb{C} \setminus \{0\}$ . Thus,  $A'$  (and hence  $A$ ) is unitarily congruent to  $A'' = \text{diag}(\tilde{B}(z_1), \dots, \tilde{B}(z_{\lfloor \frac{n}{2} \rfloor}), 1)$ , where some  $z_k$ 's can be equal to unity and the last unity block is present if  $n$  is odd. So,  $V$  is unitarily congruent to  $V'' = Q^{-1/2} A'' = D P_{\sigma_0}$ , where  $\sigma_0 = (12)(34) \dots$  and  $D = Q^{-1/2} \text{diag}(z_1, 1/z_1, z_2, 1/z_2, \dots)$ . Clearly, we have

$$Q D D^{\sigma_0} = I_n. \quad (77)$$

That is,  $D$  satisfies the second equation in (15). The first equation in (15) for  $D$  follows from the condition (9).

It remains to note that every involution  $\sigma \in \mathcal{S}_n$  that has the same number of fixed points as  $\sigma_0$  does can be constructed as  $\sigma = \tau \circ \sigma_0 \circ \tau^{-1}$  by choosing a suitable  $\tau \in \mathcal{S}_n$ . Therefore  $V$  is unitarily congruent to  $V''' = P_\tau D P_{\sigma_0} P_\tau^t = D' P_\sigma$ , where  $D' = D^\tau$ . Obviously, we have  $\text{tr } D \tilde{D} = \text{tr } D' \tilde{D}'$ . And applying the permutation  $\tau$  to (77), we verify the second equation in (15) for  $D'$ :  $I_n = Q D^\tau (D')^{\tau \circ \sigma_0} = Q D' (D')^\sigma$ .  $\square$

**Proof of Proposition 1.** Recall Jacobi's identity for submatrices (see, e.g. eq. (0.8.4.2) in [6]): if  $A$  and  $A^{-1}$  are partitioned matrices,  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ ,  $A^{-1} = \begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix}$  and the blocks  $A_{11}$  and  $A'_{11}$  are of the same size, then  $\det A'_{22} = (\det A_{11})/\det A$ . Take  $A = QW_{\mathcal{T}}$  and  $A_{11} = QV_1\bar{V}_1$ . Since  $A$  is unitary, we have  $A^{-1} = A^*$  and so  $A'_{22} = QV_2^t V_2^*$ . Invoking Jacobi's identity and taking into account that  $|\det A| = 1$ , we infer that  $|\det V_1|^2 = |\det(A_{11}/Q)| = |\det(A'_{22}/Q)| = |\det V_2|^2$ . Thus,  $|\det V_1| = |\det V_2|$  and hence  $V_1$  and  $V_2$  are either both singular or both non-singular.

Rewriting equation (24) in the form  $V_1\bar{V}_1V_2^tV_1^* = -V_2\bar{V}_1V_2^tV_2^*$  and comparing the determinants of the both sides, we infer that  $\det(\bar{V}_1V_2^t)|\det V_1|^2 = (-1)^n \det(\bar{V}_1V_2^t)|\det V_2|^2$ . Whence it follows that, if  $n$  is odd,  $V_1$  and  $V_2$  cannot be both non-singular. Taking into account that  $|\det V_1| = |\det V_2|$ , we conclude that both  $V_1$  and  $V_2$  are singular.  $\square$

**Proof of Proposition 2.** *ii)* Eq. (22) was derived from the condition  $Q^2W_{\mathcal{T}}W_{\mathcal{T}}^* = I_{2n}$ . Its counterpart derived from the equivalent condition  $Q^2W_{\mathcal{T}}^*W_{\mathcal{T}} = I_{2n}$  reads

$$V_1^tV_1^*V_1\bar{V}_1 + V_2^tV_1^*V_1\bar{V}_2 = Q^{-2}I_n. \quad (78)$$

Since  $\alpha V_1\bar{V}_1$  is unitary,  $V_1$  is non-singular and we have  $\alpha^2\bar{V}_1V_1^t = (V_1^*V_1)^{-1}$ . Using this relation, we rewrite (22) and (78) as follows:

$$(V_2V_1^{-1})(V_2V_1^{-1})^* = (\alpha^2Q^{-2} - 1)I_n, \quad (79)$$

$$(V_1^{-1}V_2)^t\overline{(V_1^{-1}V_2)} = (\alpha^2Q^{-2} - 1)I_n. \quad (80)$$

Taking Proposition 1 into account, we infer that the l.h.s. of (79) and (80) are positive definite matrices and their determinants are equal to unity. Therefore,  $\alpha^2Q^{-2} - 1 = 1$  and thus  $\alpha = \sqrt{2}Q$ . Furthermore, (79) and (80) imply, respectively, that  $V_2 = g'V_1$  and  $V_2 = V_1g$  where  $g$  and  $g'$  are unitary.

*i)* As a consequence of *ii)*, we have  $\alpha V_1\bar{V}_2 = \alpha V_1\bar{V}_1\bar{g}$ ,  $\alpha V_2\bar{V}_1 = \alpha g'V_1\bar{V}_1$ ,  $\alpha V_2\bar{V}_2 = \alpha g'V_1\bar{V}_1\bar{g}$ , and so all these matrices are unitary.

*iii)* Another consequence of *ii)* is  $\bar{V}_2V_2^t = \bar{V}_1V_1^t$  and hence  $\text{tr } V_2V_2^* = \text{tr } V_1V_1^* = 1$ . Multiplying (24) with  $V_1^{-1}$  from the left and with  $(V_1^*)^{-1}$  from the right, taking trace, and taking into account that  $V_1^{-1}V_2 = g$  is unitary, we conclude that  $\text{tr } \bar{V}_1V_2^t = 0$ .

*iv)* The inequality on  $Q$  is implied by Proposition 7 from [2] for  $r = 2$ .  $\square$

**Proof of Theorem 3.** For  $V_1, V_2$  given by (27), the corresponding matrix  $W_{\mathcal{T}}$  can be brought to a block diagonal form by permutations of its block-rows and block-columns. Indeed, let  $P_1$  and  $P_2$  be the permutation matrices corresponding to the permutations  $\{123456\} \rightarrow \{143625\}$  and  $\{123456\} \rightarrow \{134625\}$ , respectively. Then we have

$$(P_1 \otimes I_p)W_{\mathcal{T}}(P_2^t \otimes I_p) = \begin{pmatrix} W_1 & 0 \\ 0 & \bar{W}_2 \end{pmatrix}, \quad (81)$$

where  $W_1 \in M_{4p}$  and  $W_2 \in M_{2p}$  are given by

$$\alpha_1\alpha_2 W_1 = H_1 \otimes_{2 \times 2} H_2 \quad (82)$$

and

$$W_2 = \begin{pmatrix} G_{11}F_{11} + G_{12}F_{21} & G_{21}F_{11} + G_{22}F_{21} \\ G_{11}F_{12} + G_{12}F_{22} & G_{21}F_{12} + G_{22}F_{22} \end{pmatrix}. \quad (83)$$

In (82), the Kronecker product is understood as that for  $2 \times 2$  matrices  $H_1, H_2$  that have noncommuting entries  $\alpha_1 F_{ij}, \alpha_2 G_{ij}$ . In other words, we have

$$W_1 = \sum_{a,b,c,d=1}^2 E_{ab} \otimes E_{cd} \otimes F_{ab} G_{cd}, \quad (84)$$

where  $E_{ab}$  are the basis  $2 \times 2$  matrices, i.e.  $(E_{ab})_{ij} = \delta_{ai}\delta_{bj}$ .

Equation (81) implies that  $QW_{\mathcal{T}}$  is unitary iff  $QW_1$  and  $QW_2$  are unitary. By (84), we have

$$W_1 W_1^* = \sum_{a,b,c,d,i,j=1}^2 E_{ab} \otimes E_{cd} \otimes F_{ai} G_{cj} G_{dj}^* F_{bi}^*. \quad (85)$$

Therefore, if  $H_1, H_2$  are unitary, that is if the following relations hold:

$$\alpha_1^2 \sum_{b=1}^2 F_{ab} F_{cb}^* = \alpha_1^2 \sum_{b=1}^2 F_{ba}^* F_{bc} = \delta_{ac} I_p = \alpha_2^2 \sum_{b=1}^2 G_{ab} G_{cb}^* = \alpha_2^2 \sum_{b=1}^2 G_{ba}^* G_{bc}, \quad (86)$$

we infer from (85) that  $\alpha_1 \alpha_2 W_1$  is unitary.

Next, if relation (30) holds, we can rewrite  $W_2$  given by (83) in the following form:

$$\alpha_1 \alpha_2 W_2 = S H_2 H_1 \bar{S}, \quad S = \text{diag}(\zeta^{\frac{1}{2}}, \bar{\zeta}^{\frac{1}{2}}) \otimes I_p. \quad (87)$$

Whence it is evident that  $\alpha_1 \alpha_2 W_2$  is unitary if  $H_1, H_2$  are unitary.

Finally, we note that relations (20) for  $V_1, V_2$  given by (27) acquire the following form:

$$\text{tr } V_i V_j^* = \sum_{a=1}^2 \text{tr } F_{ai} F_{aj}^* + \sum_{a=1}^2 \text{tr } \bar{G}_{ia} G_{ja}^t = \delta_{ij}. \quad (88)$$

Taking relations (86) into account, we see that (88) holds providing that  $H_1, H_2$  are unitary and condition (28) is satisfied.  $\square$

**Proof of Proposition 3.** *i)* We have  $H_1 = U, H_2 = U^*$ . The l.h.s. and the r.h.s. of (30) are equal, respectively, to the (12) and (21) blocks of  $(U^*U)$  and hence they vanish identically.

*ii)* It is straightforward to verify that  $H_1, H_2$  are unitary and that both sides of (30) vanish identically.  $\square$

**Proof of Proposition 4.** It is straightforward to verify that  $H_1, H_2$  are unitary providing that relations (43) hold. Further, we have  $G_{11}F_{12} + G_{12}F_{22} = MZ_1^{\sigma_2 \circ \sigma_1} P_{\sigma_2} P_{\sigma_1}$  and  $G_{21}F_{12} + G_{22}F_{22} = -Z_2 \bar{M} P_{\sigma_2} P_{\sigma_1}$ . Therefore condition (44) implies equality (30).  $\square$



**Proof of Lemma 2.** For the sake of brevity, if  $D$  is a diagonal matrix, we will write for its diagonal entries  $D_i$  instead of  $D_{ii}$ . Recall that  $D_1, D_2$  are non-singular.

If  $\sigma_1, \sigma_2$  does not satisfy (52), then  $\sigma' \neq \sigma''$  and so there exist  $i, j$  such that  $(P_{\sigma'})_{ij} = 1$  but  $(P_{\sigma''})_{ij} = 0$  and thus the  $(ij)$  matrix entry of the l.h.s. of (51) cannot vanish.

Suppose that  $\sigma_1, \sigma_2$  satisfy (52) but  $\sigma_1(i) = \sigma_2(i) = i$  for some  $i$ . Then we have  $(P_{\sigma'})_{ii} = (P_{\sigma''})_{ii} = 1$ . Therefore the  $(ii)$  matrix entry of the l.h.s. of (51) is  $M_{ii} \equiv (|(D_1)_i|^2 + |(D_2)_i|^2)(\bar{D}_1)_i(D_2)_i$  and so it cannot vanish.

If  $\sigma_1$  and  $\sigma_2$  are involutions or they commute, then  $\sigma_1, \sigma_2$  satisfy (52) and  $\sigma' = \sigma'' = \sigma_2^{-1} \circ \sigma_1$ . Suppose that  $(\sigma_2^{-1} \circ \sigma_1)(i) = i$  for some  $i$ . Then  $(P_{\sigma'})_{ii} = (P_{\sigma''})_{ii} = 1$ . Therefore the  $(ii)$  matrix entry of the l.h.s. of (51) is  $M_{ii} \equiv |(D_1)_i|^2(\bar{D}_1)_{\sigma_1^{-1}(i)}(D_2)_{\sigma_1^{-1}(i)} + |(D_2)_i|^2(\bar{D}_1)_{\sigma_2^{-1}(i)}(D_2)_{\sigma_2^{-1}(i)}$ . Note that, for commuting  $\sigma_1$  and  $\sigma_2$ , equality  $(\sigma_2^{-1} \circ \sigma_1)(i) = i$  implies that  $\sigma_2^{-1}(i) = \sigma_1^{-1}(i)$ . The same is true if  $\sigma_1$  and  $\sigma_2$  are involutions. Therefore,  $M_{ii} = (|(D_1)_i|^2 + |(D_2)_i|^2)(\bar{D}_1)_{\sigma_1^{-1}(i)}(D_2)_{\sigma_1^{-1}(i)}$  which cannot vanish.  $\square$

**Proof of Proposition 5.** *i)* The group  $\mathcal{S}_4$  splits into five nonintersecting conjugacy classes,  $\mathcal{O}_i$ ,  $i = 0, \dots, 4$ . For every two elements  $\sigma_1, \sigma_2 \in \mathcal{O}_i$ , there exists  $\tau \in \mathcal{S}_4$  such that  $\sigma_2 = \tau^{-1} \circ \sigma_1 \circ \tau$ .  $\mathcal{O}_0$  contains only  $\sigma = id$ .  $\mathcal{O}_1$  contains six involutions that have two fixed points, e.g.  $\sigma = (1)(23)(4)$ .  $\mathcal{O}_2$  contains eight elements that have one fixed point and are of order three, e.g.  $\sigma = (123)(4)$ .  $\mathcal{O}_3$  contains six elements that have no fixed points and are of order four, e.g.  $\sigma = (1234)$ .  $\mathcal{O}_4$  contains three involutions that have no fixed points, e.g.  $\sigma = (12)(34)$ . Without a loss of generality, we will take the mentioned above representatives of each conjugacy class  $\mathcal{O}_i$  as  $\sigma_1$  and will search for all inequivalent admissible pairs  $\sigma_1, \sigma_2$ , where  $\sigma_2 \in \mathcal{O}_j$ ,  $j \geq i$ .

For  $\sigma_1 = id$ ,  $\sigma_2$  must have no fixed points. We can take as  $\sigma_2$  the mentioned above representatives of  $\mathcal{O}_3$  and  $\mathcal{O}_4$ .

For  $\sigma_1 = (1)(23)(4)$ , the only suitable  $\sigma_2$  from  $\mathcal{O}_1$  is  $(14)(2)(3)$  because  $\sigma_2^{-1} \circ \sigma_1$  must have no fixed points. Further, note that  $\sigma_2$  cannot be from  $\mathcal{O}_2$  because, in this case, equation (52) would imply that  $\sigma_1 \asymp \sigma_2^{-2} = \sigma_2$ . However, the commutant of every  $\sigma \in \mathcal{O}_2$  consists only of  $id, \sigma, \sigma^{-1}$ . The suitable elements from  $\mathcal{O}_3$  are  $\sigma_2 = (1342)$ ,  $\sigma_2 = (1243)$  and these from  $\mathcal{O}_4$  are  $\sigma_2 = (12)(34)$ ,  $\sigma_2 = (13)(24)$ . In the each case, the corresponding admissible pairs are equivalent by the transformation (46) with  $\tau = \sigma_1$ .

The consideration for  $\sigma_1 \in \mathcal{O}_2, \mathcal{O}_3$  is similar and we omit its details. Finally, for  $\sigma_1 = (12)(34)$ ,  $\sigma_2$  can be either of the other two elements from  $\mathcal{O}_4$ . The corresponding admissible pairs are equivalent by the transformation (46) with  $\tau = (1)(2)(34)$ .

*ii)* We have  $4D_i\bar{D}_i = I_4$  and hence (47) is satisfied and (49)–(50) hold for  $Q^2 = 8$ . Note that  $\sigma_2^{-1} \circ \sigma_1$  has no fixed points in all the cases except h). So, (48) is satisfied trivially except for the case h), where we have  $\text{tr}(D_1\bar{D}_2P_{\sigma_1}P_{\sigma_2}^t) = e^{i\pi(u_2-v_2)} + e^{i\pi(u_4-v_4)}$ . Note also that  $\sigma_1, \sigma_2$  satisfy the hypotheses of Theorem 4 in all the cases except h) and f). Therefore, in all these cases, it is sufficient to find  $\vec{u}, \vec{v}$  that fulfil condition (60). It is straightforward to check that suitable pairs of vectors can be chosen as follows:  $\vec{u} = 0$  for all the cases except c) and

$$\begin{aligned} a), b) : 4\vec{v} &= (1, -1, 1, -1); & c) : 4\vec{u} &= (0, 1, -1, 0), \quad 4\vec{v} = (1, 0, 0, -1); \\ d), e) : 2\vec{v} &= (1, 0, 0, 1); & f), g), h) : \vec{v} &= (1, 1, 0, 0); & i), j) : \vec{v} &= (1, 0, 0, 0). \end{aligned}$$

In the cases h) and f), one has to verify relation (51) by inspection.  $\square$

**Proof of Proposition 6.**  $\sigma_1, \sigma_2$  is an admissible pair because  $\sigma_1 \asymp \sigma_2$  and  $\sigma_2^{-1} \circ \sigma_1$  has no fixed points. The latter property implies also that (48) is satisfied trivially. Set  $D_1 = \frac{1}{\sqrt{n}} \text{diag}(e^{i\pi u_1}, \dots, e^{i\pi u_n})$  and  $D_2 = \frac{1}{\sqrt{n}} \text{diag}(e^{i\pi v_1}, \dots, e^{i\pi v_n})$ ,  $\vec{u}, \vec{v} \in \mathbb{R}^n$ . Then  $nD_i \bar{D}_i = I_n$  and hence (47) is satisfied and (49)–(50) hold for  $Q^2 = n^2/2$ . Note that  $\sigma_1, \sigma_2$  satisfy the hypotheses of Theorem 4, hence it is sufficient to find  $\vec{u}, \vec{v}$  that fulfil condition (60). Consider vectors  $\vec{y}_i \in \text{Ker}(I_n + P_{\sigma_i})$ ,  $i = 1, 2$  such that  $(\vec{y}_i)_k = 0$  if  $k$  is a fixed point of  $\sigma_i$  and  $(\vec{y}_i)_k = \pm 1$  otherwise. Clearly, there is an equal amount of  $+1$  and  $-1$  among the components of  $\vec{y}_i$  corresponding to each cycle in  $\sigma_i$ . Since  $\sigma_1$  and  $\sigma_2$  have complementary sets of fixed points, we have a)  $(\vec{y}_1 + \vec{y}_2)_k = \pm 1$  for all  $k$ ; b)  $P_{\sigma_2} \vec{y}_1 = \vec{y}_1$ ,  $P_{\sigma_1} \vec{y}_2 = \vec{y}_2$ , so that  $A\vec{y}_1 = 4\vec{y}_1$ ,  $B\vec{y}_2 = 4\vec{y}_2$ . Therefore  $\vec{u} = \frac{1}{4}\vec{y}_1$  and  $\vec{v} = \frac{1}{4}\vec{y}_2$  fulfil condition (60).  $\square$

**Proof of Theorem 4.** For the brevity of notations, let  $e^{\vec{y}}$ , where  $\vec{y} \in \mathbb{C}^n$ , stand for the diagonal matrix  $\text{diag}(e^{y_1}, \dots, e^{y_n})$ . Then, for  $D_1, D_2$  given by (61), we have  $D_i \bar{D}_i = \mu^{-2} e^{2\vec{x}}$  and therefore both equations (49)–(50) are equivalent to the following one:

$$\mu^{-2} e^{2(I_n + P_{\sigma_1})\vec{x}} + \mu^{-2} e^{2(I_n + P_{\sigma_2})\vec{x}} = Q^{-2} I_n. \quad (89)$$

If (59) is satisfied, then (89) holds and we have  $Q^2 = \mu^2/2$ .

Since  $\sigma_1, \sigma_2$  satisfy (53), we have to verify that (55) holds. Substituting (61) into (55), we obtain the following equation:

$$e^{(P_{\sigma_1} + P_{\sigma_2})(I_n + P_{\sigma_1})\vec{x} + i\pi(A - P_{\sigma_2}^2)\vec{u} + i\pi P_{\sigma_1}^2 \vec{v}} = -e^{(P_{\sigma_1} + P_{\sigma_2})(I_n + P_{\sigma_2})\vec{x} - i\pi(B - P_{\sigma_1}^2)\vec{v} - i\pi P_{\sigma_2}^2 \vec{u}}, \quad (90)$$

where  $A, B$  are given by (58). If (59) is satisfied, then (90) is equivalent to equation  $e^{i\pi \vec{w}} = -I_n$  which implies that all the components of  $\vec{w}$  must be odd integers.

It remains to note that relations (47)–(48) are satisfied thanks to the choice of  $\mu$  in (61) and the condition that  $\sigma_2^{-1} \circ \sigma_1$  has no fixed points.  $\square$

**Proof of Proposition 7.**  $\zeta, z_1, z_2 \in \mathbb{C}$  satisfying (64) can be parametrized as follows:  $\zeta = e^{i\pi\phi}$ ,  $z_1 = \mu^{-1} e^{x+i\pi\alpha}$ ,  $z_2 = \mu^{-1} e^{-x+i\pi\beta}$ , where  $x, \mu, \alpha, \beta \in \mathbb{R}$  and  $\mu^2 = n \cosh(2x)$ . Therefore,  $D_1, D_2$  are given by (61), where  $\vec{x} = (x, -x, x, -x, \dots)$ ,  $\vec{u} = (\alpha, \beta, \alpha, \beta, \dots)$ , and  $\vec{v} = \vec{u} + \vec{\rho}$ ,  $\vec{\rho} = (0, \phi + 1, 0, \phi, \dots)$ . For  $n = 4l$  and  $\sigma_1, \sigma_2$  given by (56) or (57), such  $\vec{x}$  satisfies (59) and, furthermore, we have  $(P_{\sigma_1} - P_{\sigma_2})\vec{u} = 0$  and  $\vec{\rho}' \equiv (P_{\sigma_1} - P_{\sigma_2})\vec{\rho} = (-1, 0, 1, 0, \dots)$ . Thus, for  $A, B$  given by (58), we have  $\vec{w} = A\vec{u} + B\vec{v} = B\vec{\rho} = (I_n + P_{\sigma_1})\vec{\rho}'$ . For the either choice of  $\sigma_1$ , all the components of  $\vec{w}$  are odd integers and so the hypotheses of Theorem 4 are satisfied.  $\square$

**Proof of Proposition 8.** For  $D_1, D_2$  given by (69), we have  $D_1 \bar{D}_1 D_1^{\sigma_1} \bar{D}_1^{\sigma_1} = D_0$ ,  $D_2 \bar{D}_2 D_1^{\sigma_2} \bar{D}_1^{\sigma_2} = P_{\sigma_2} D_0$ ,  $D_1 \bar{D}_1 D_2^{\sigma_1} \bar{D}_2^{\sigma_1} = P_{\sigma_1} D_0$ , and  $D_2 \bar{D}_2 D_2^{\sigma_2} \bar{D}_2^{\sigma_2} = P_{\sigma_1^{-1}} D_0$ , where  $D_0 \equiv |z_1|^2 \text{diag}_4(|z_3|^2, |z_2|^2, |z_2|^2, |z_3|^2)$ . Since  $(I_n + P_{\sigma_2})D_0 = (P_{\sigma_1} + P_{\sigma_1^{-1}})D_0 = Q^{-2} I_n$ , where  $Q$  is given by (71), we conclude that equations (49)–(50) hold. Since  $\sigma_1, \sigma_2$  satisfy (53), it suffices to verify (55). A direct computation yields  $D_1^{\sigma_2} \bar{D}_1^{\sigma_2 \circ \sigma_1} D_2^{\sigma_1 \circ \sigma_1} \bar{D}_1^{\sigma_1} = |z_1|^2 \text{diag}_4(z_2 \bar{z}_3, \zeta \bar{z}_2 \bar{z}_3, \bar{z}_2 z_3, -\zeta \bar{z}_2 \bar{z}_3) = -D_2^{\sigma_2} \bar{D}_1^{\sigma_2 \circ \sigma_2} D_2^{\sigma_1 \circ \sigma_2} \bar{D}_1^{\sigma_1}$ , so that (55) holds.

It remains to note that (47) holds thanks to the condition (70) whereas (48) is equivalent to the condition  $\sum_{k=1}^{2l} (D_1 \bar{D}_2)_{2k, 2k} = 0$  which also holds as seen from (69).  $\square$

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